VARIETIES WITH DEGENERATE GAUSS MAPS WITH MULTIPLE FOCI AND TWISTED CONES

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Abstract. The authors study in detail new types of varieties with degenerate Gauss maps: varieties with multiple foci and their particular case, the so-called twisted cones. They prove an existence theorem for twisted cones and describe their structure.

0   Introduction

A smooth n-dimensional variety X of a projective space P^N is called *tangentially degenerate* or *a variety with a degenerate Gauss map* if the rank of its Gauss mapping $\gamma : X \to G(n, N)$ is less than n, $0 \leq r = \operatorname{rank} \gamma < n$. Here $x \in X$, $\gamma(x) = T_x(X)$, and $T_x(X)$ is the tangent subspace to X at x considered as an n-dimensional projective space P^n. The number r is also called the *rank* of X, $r = \operatorname{rank} X$. The case $r = 0$ is trivial one: it gives just an n-plane.

Let $X \subset P^N$ be an n-dimensional smooth variety with a degenerate Gauss map. Suppose that $0 < \operatorname{rank} \gamma = r < n$. Denote by L a leaf of this map, $L = \gamma^{-1}(T_x) \subset X$; $\dim L = n - r = l$. The number l is called the Gauss defect of the variety X (see [FP 01], p. 89, or [L 99], p. 52) or the index of relative nullity of X (see [CK 52]).

A variety with a degenerate Gauss map of rank r foliates into their leaves L of dimension l, along which the tangent subspace $T_x(X)$ is fixed. The foliation on X with leaves L is called the *Monge–Ampère foliation* (see, for example, [D 89] or [I 98, 99b]).

However, unlike a traditional definition of the foliation, the leaves of the Monge–Ampère foliation have singularities. This is a reason that in general its leaves are not diffeomorphic to a standard leaf. We assume that singular points belong to the leaf L, and hence the leaf is an l-dimensional subspace of the space P^N.

The tangent subspace $T_x(X)$ is fixed when a point x moves along regular points of L. This is the reason that we denote it by T_L, $L \subset T_L$. A pair (L, T_L) on X depends on r parameters.

The varieties of rank $r < n$ are multidimensional analogues of developable surfaces of a three-dimensional Euclidean space. They were first considered by É. Cartan [C 16] in connection with his study of metric deformations of hypersurfaces, and in [C 19] in connection with his study of manifolds of constant curvature. Recently varieties with degenerate Gauss maps of rank $r < n$ are

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intensively studied both from the projective point of view and the Euclidean point of view.

The main results on the geometry of varieties with degenerate Gauss maps and further references can be found in Chapter 4 of the book [AG 93] and in the recently published paper [AG 01b].

Griffiths and Harris [GH 79] (Section 2, pp. 383–393) considered varieties with degenerate Gauss maps from the point of view of algebraic geometry. Following [GH 79], Landsberg published notes [L 99] which are in some sense an update to the paper [GH 79]. Section 5 (pp. 47–50) of these notes is devoted to varieties with degenerate Gauss maps.

In particular, in [GH 79] Griffiths and Harris presented a structure theorem for varieties with degenerate Gauss maps. They asserted that such varieties are "built up from cones and developable varieties" (see [GH 79], p. 392). They gave a proof of this assertion in the case $n = 2$. Their result appears to be complete for varieties whose Gauss maps have one-dimensional fibers. However, their result is incomplete for tangentially degenerate hypersurfaces whose Gauss maps have fibers of dimension greater than one. In the recently paper [AG 01b], Akivis and Goldberg showed that there exist hypersurfaces which cannot be built out of cones and osculating varieties.

In the current paper we define and study new types of varieties with degenerate Gauss maps: varieties with multiple foci and their particular case, the so-called twisted cones.

1 Basic equations of a hypersurface of rank r with r-multiple focus hyperplanes

In our recently published paper [AG 02] in a projective space P^N, we considered varieties X with degenerate Gauss maps of dimension n and rank r with the following two properties:

(i) Their focus hypersurfaces F_L degenerate into r-fold hyperplanes.

(ii) Their system of second fundamental forms possesses at least two forms, whose λ-equation has r distinct roots.

We have proved that such varieties X are cones in the space P^N with a vertex of dimension $l - 1$, where $l = n - r$.

In this paper we also consider varieties X with degenerate Gauss maps of dimension n and rank r with r-fold focus hyperplanes but we assume that all their second fundamental forms are proportional, i.e., for each pair of second fundamental forms of X, their λ-equation has r-multiple eigenvalues.

Since we assume $r \geq 2$, Segre's theorem (see [AG 93], Theorem 2.2, p. 55) implies that such varieties are hypersurfaces in a subspace P^{n+1}. We shall prove that such hypersurfaces can differ from cones.

Consider a hypersurface X with a degenerate Gauss map of dimension n and rank r whose focus hypersurfaces F_L are r-fold hyperplanes of dimension

$l - 1$, where $l = n - r$ is the dimension of the Monge–Ampère foliation on X. We associate a family of moving frames $\{A_u\}$, $u = 0, 1, \ldots, n + 1$, with X in such a way that the point A_0 is a regular point of a generator L, the points A_a, $a = 1, \ldots, l$, belong to the r-fold focus hyperplane F_L, the points A_p, $p = l + 1, \ldots, n$, lie in the tangent hyperplane $T_L(X)$, and the point A_{n+1} is situated outside of this hyperplane.

The equations of infinitesimal displacement of the moving frame $\{A_u\}$ are

$$dA_u = \omega_u^v A_v, \quad u, v = 0, 1, \ldots, n + 1, \tag{1}$$

where ω_u^v are 1-forms satisfying the structure equations of the projective space P^N:

$$d\omega_u^v = \omega_u^w \wedge \omega_w^v, \quad u, v, w = 0, 1, \ldots, n + 1. \tag{2}$$

As a result of the specialization of the moving frame mentioned above, we obtain the following basic equations of the variety X:

$$\omega_0^{n+1} = 0, \quad \omega_a^{n+1} = 0, \quad a = 1, \ldots, l, \tag{3}$$

$$\omega_p^{n+1} = b_{pq} \omega^q, \quad \omega_a^p = c_{aq}^p \omega^q, \tag{4}$$

and

$$b_{sq} c_{ap}^s = b_{sp} c_{aq}^s; \tag{5}$$

here $\omega^q := \omega_0^q$ are the basis forms of the variety X, and $B = (b_{pq})$ is a nondegenerate symmetric $(r \times r)-$ matrix (see [AG 93], Section 4.1).

Denote by C_a the $(r \times r)$-matrix occurring in equations (4):

$$C_a = (c_{aq}^p).$$

If we use the identity matrix $C_0 = (\delta_q^p)$ and the index $i = 0, 1, \ldots, l$ (i.e., $\{i\} = \{0, a\}$), then equations $b_{pq} = b_{qp}$ and (5) can be combined and written as follows:

$$(BC_i)^T = (BC_i), \tag{6}$$

i.e., the matrices

$$H_i = BC_i = (b_{qs} c_{ip}^s)$$

are symmetric.

Since the points A_a, $a = 1, \ldots, l$, belong to the r-fold focus $(l-1)$-plane F_L, equation of F_L is

$$(x^0)^r = 0.$$

But in the general case the focus hypersurface F_L of the generator L is determined by the equation

$$\det \left(\delta_q^p x^0 + c_{aq}^p x^a \right) = 0$$

(see equation (4.19) on p. 117 of the book [AG 93]). Thus, we have

$$\det(\delta_q^p x^0 + c_{aq}^p x^a) = (x^0)^r$$

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It follows that each of the matrices C_a has an r-multiple eigenvalue 0, and as a result, each of these matrices is nilpotent. We assume that each of the matrices C_a has the form

$$C_a = (c_{aq}^p), \text{ where } c_{aq}^p = 0 \text{ for } p \geq q. \tag{7}$$

Thus, rank $C_a \leq r - 1$. It follows that all matrices C_a are nilpotent. Denote by r_1 the maximal rank of matrices from the bundle $C = x^a C_a$, $r_1 \leq r - 1$.

It is obvious that this form is sufficient for all F_L to be r-fold hyperplanes. Wu and Zheng [WZ 02] (see also Piontkowski [P 01, 02]) proved this for the ranks $r = 2, 3, 4$ and different values of the maximum rank r_1 of matrices of the bundle $x^a C_a$. However, Wu and Zheng in [WZ 02] gave also a counterexample which proves that the form (7) is not necessary for all F_L to be r-fold hyperplanes.

A single second fundamental form of X at its regular point $x = A_0$ can be written as

$$\Phi_0 = b_{pq} \omega^p \omega^q.$$

This form is of rank r. At the singular points A_a belonging to an r-multiple focus hyperplane F_L, the second fundamental form of the hypersurface X has the form

$$\Phi_a = b_{ps} c_{aq}^s \omega^p \omega^q, \tag{8}$$

where $(b_{ps} c_{aq}^s)$ is a symmetric matrix. The maximal rank of matrices from the bundle $\Phi = x^a \Phi_a$ is also equal to $r_1 < r - 1$.

2 Hypersurfaces with degenerate Gauss maps of rank r with a one-dimensional Monge–Ampère foliation and r-multiple foci

Let $A_0 A_1$ be a leaf of the Monge–Ampére foliation, let A_0 be a regular point of this foliation, and let A_1 be its r-multiple focus. Then in equations (6), we have $a, b = 1$; $p, q = 2, \ldots, n$, and these equations become

$$\omega_p^{n+1} = b_{pq} \omega^q, \quad \omega_1^p = c_q^p \omega^q. \tag{9}$$

By our assumption (7), the matrix $C = (c_q^p)$ has the form

$$C = \begin{pmatrix} 0 & c_3^2 & \ldots & c_n^2 \\ \cdots\cdots\cdots\cdots \\ 0 & 0 & \ldots & c_n^{n-1} \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \tag{10}$$

where the coefficients $c_{p+1}^p \neq 0$. As to the matrix $B = (b_{pq})$, by the relation

$$BC = CB \tag{11}$$

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(cf. (6)), this matrix has the form

$$
B = \begin{pmatrix}
0 & \ldots & 0 & b_{2,n} \\
0 & \ldots & b_{3,n-1} & b_{3,n} \\
\multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\
b_{n,2} & \ldots & b_{n,n-1} & b_{nn}
\end{pmatrix},
\tag{12}
$$

and rank $C = n - 2$, rank $B = n - 1$. In addition, by (11), the entries of the matrices B and C are connected by certain bilinear relations implied by (11).

By (9), (10), and (12), on the hypersurface X, we have the equation

$$
\omega_1^n = 0.
\tag{13}
$$

Since on the hypersurface X also the equations (3) hold, the differentials of the points A_0 and A_1 take the form

$$
\begin{aligned}
dA_0 &= \omega_0^0 A_0 + \omega_0^1 A_1 + \omega_0^2 A_2 + \ldots + \omega_0^{n-1} A_{n-1} + \omega_0^n A_n, \\
dA_1 &= \omega_1^0 A_0 + \omega_1^1 A_1 + \omega_1^2 A_2 + \ldots + \omega_1^{n-1} A_{n-1}.
\end{aligned}
\tag{14}
$$

In equations (14), the forms $\omega_1^2, \omega_1^3, \ldots, \omega_1^{n-1}$ are linearly independent, and by (9) and (10), they are expressed in terms of the basis forms $\omega^3, \ldots, \omega^n$ only. The following cases can occur:

1) The 1-form ω_1^0 is independent of the forms $\omega^3, \ldots, \omega^n$, and hence also of the forms $\omega_1^2, \ldots, \omega_1^{n-1}$. In this case, the r-multiple focus A_1 of the rectilinear generator L describes a focus variety G of dimension $r = n-1$. The variety G is of codimension two in the space P^{n+1} in which the hypersurface X is embedded. The tangent subspace $T_{A_1}(G)$ is defined by the points $A_1, A_0, A_2, \ldots, A_{n-1}$. At the point A_1, the variety G has two independent second fundamental forms. We can determine these two forms by finding the second differential of the point A_1:

$$
d^2 A_1 \equiv \omega_1^p \omega_p^n A_n + \omega_1^p \omega_p^{n+1} A_{n+1} \pmod{T_{A_1}(G)}.
$$

Thus, we have
$$
\Phi_1^n = \omega_1^p \omega_p^n, \quad \Phi_1^{n+1} = \omega_1^p \omega_p^{n+1}.
$$

The second of these forms coincides with the second fundamental form Φ_1 of the hypersurface X at the point A_1. By (10), if $\omega^3 = \ldots = \omega^n = 0$, the 1-forms $\omega_1^p = 0$. Hence the quadratic forms Φ_1^n and Φ_1^{n+1} vanish on the focal variety G. Thus the direction $A_1 \wedge A_0$ is an asymptotic direction on G.

2) The 1-form ω_1^0 is a linear combination of the forms $\omega_1^2, \ldots, \omega_1^{n-1}$, and hence also of the forms $\omega^3, \ldots, \omega^n$. In this case, the focus A_1 of the rectilinear generator L describes a focus variety G of dimension $n-2$, and its tangent subspace $T_{A_1}(G)$ is a hyperplane in the space $A_0 \wedge A_1 \wedge A_2 \wedge \ldots \wedge A_{n-1}$. For

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$\omega_1^2 = \ldots = \omega_1^{n-1} = 0$, the point A_1 is fixed, and the straight line $L = A_1 A_0$ describes a two-dimensional cone with vertex A_1. This cone is called the *fiber cone*. The hypersurface X foliates into an $(n-2)$-parameter family of such fiber cones. It is called a *twisted cone with rectilinear generators*.

Later on, in Section 3, for $n = 3$ we will prove that a fiber cone is a pencil of straight lines. Most likely this is true for any n.

3) Suppose that an $(n-2)$-dimensional focus variety G of the hypersurface X belongs to a hyperplane P^n of the space P^{n+1}. We can take this hyperplane as the hyperplane at infinity P_∞^n of the space P^{n+1}. As a result, the space P^{n+1} becomes an affine space A^{n+1}. In this case, the hypersurface X becomes also a *twisted cylinder* in A^{n+1}, which foliates into an $(n-2)$-parameter family of two-dimensional cylinders with rectilinear generators. The hypersurface X with a degenerate Gauss map is not a cylinder in A^{n+1} and does not have singularities in this space. Thus, this hypersurface is an affinely complete hypersurface in A^{n+1}, which is not a cylinder. An example of such a hypersurface in the space A^{n+1} was considered by Sacksteder and Bourgain (see [S 60], [W 95], [I 98, 99a, b], and [AG 01a]).

Note also that hypersurfaces with degenerate Gauss maps in the space P^{n+1} considered in this section are *lightlike hypersurfaces* which were studied in detail in the papers [AG 98a, b; 99a, b] by Akivis and Goldberg.

3 Hypersurfaces with Degenerate Gauss Maps with Double Foci on Their Rectilinear Generators in the Space P^4

As an example, we consider hypersurfaces X with degenerate Gauss maps of rank $r = 2$ in the space P^4 that have a double focus F on each rectilinear generator L. On a rectilinear generator $L = A_0 \wedge A_1$ of X, there is a single (double) focus F_L. With respect to a first-order frame, the basic equations of X are

$$\omega_0^4 = 0, \quad \omega_1^4 = 0. \tag{15}$$

The basis forms of X are ω_0^2 and ω_0^3. By (4) and (12), with respect to a second-order frame, we have the following equations

$$\begin{cases} \omega_2^4 = \phantom{b_{32}\omega_0^2 +} b_{23}\omega_0^3, & \omega_1^2 = c_3^2\omega_0^3, \\ \omega_3^4 = b_{32}\omega_0^2 + b_{33}\omega_0^3, & \omega_1^3 = 0, \end{cases} \tag{16}$$

where $b_{23} = b_{32} \neq 0$ and $c_3^2 \neq 0$. As a result, the matrices B and C take the form

$$B = \begin{pmatrix} 0 & b_{23} \\ b_{23} & b_{33} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c_3^2 \\ 0 & 0 \end{pmatrix}.$$

The differentials of the points A_0 and A_1 are

$$dA_0 = \omega_0^0 A_0 + \omega_0^1 A_1 + \omega_0^2 A_2 + \omega_0^3 A_3,$$
$$dA_1 = \omega_1^0 A_0 + \omega_1^1 A_1 + \omega_1^2 A_2.$$

(cf. (14)). The point $A_1 = F_L$ is a single focus of a rectilinear generator L.

Exterior differentiation of equations (16) gives the following exterior quadratic equations:

$$-2b_{23}\omega_2^3 \wedge \omega_0^2 + \Delta b_{23} \wedge \omega_0^3 = 0, \tag{17}$$

$$\Delta b_{23} \wedge \omega_0^2 + \Delta b_{33} \wedge \omega_0^3 = 0, \tag{18}$$

$$-(\omega_1^0 + c_3^2\omega_2^3) \wedge \omega_0^2 + \Delta c_3^2 \wedge \omega_0^3 = 0, \tag{19}$$

$$(\omega_1^0 - c_3^2\omega_2^3) \wedge \omega_0^3 = 0, \tag{20}$$

where

$$\Delta b_{23} = db_{23} + b_{23}(\omega_0^0 - \omega_2^2 - \omega_3^3 + \omega_4^4) - b_{33}\omega_2^3,$$
$$\Delta b_{33} = db_{33} + b_{33}(\omega_0^0 - 2\omega_3^3 + \omega_4^4) + b_{32}c_3^2\omega_0^1 - b_{32}\omega_3^2,$$
$$\Delta c_3^2 = dc_3^2 + c_3^2(\omega_0^0 - \omega_1^1 + \omega_2^2 - \omega_3^3).$$

From equations (17) and (20), it follows that the forms ω_2^3 and ω_1^0 are linear combinations of the basis forms ω_0^2 and ω_0^3. Three cases are possible:

1) $\omega_1^0 \wedge \omega_0^3 \neq 0$. Since by (16), this implies that $\omega_1^0 \wedge \omega_1^2 \neq 0$, it follows that the focus A_1 describes a two-dimensional focal surface G^2. The tangent plane to G^2 at the point A_1 is $T_{A_1}(G) = A_1 \wedge A_0 \wedge A_2$, and the straight line $L = A_0 A_1$ is tangent to G^2 at A_1.

2)
$$\omega_1^0 \wedge \omega_0^3 = 0. \tag{21}$$

In this case, the point A_1 describes a focal line G^1, and the straight line $L = A_0 A_1$ intersects this line G^1 at the point A_1. The hypersurface X foliates into a one-parameter family of two-dimensional cones and is a twisted cone.

3) The osculating hyperplane of the curve G^1 described in 2) is fixed.

We consider these three cases in detail.

1) We prove an existence theorem for this case applying the Cartan test (see, for example, [BCGGG 91]).

Theorem 1. *Hypersurfaces X of rank two in the space P^4, for which the single focus of a rectilinear generator L describes a two-dimensional surface, exist, and the general solution of the system defining such hypersurfaces depends on one function of two variables. The direction $A_1 A_0$ is an asymptotic direction on the surfaces G^2, and the hypersurface X is formed by the asymptotic tangents to the surfaces G^2.*

Proof. On a hypersurface in question, the inequality $\omega_0^1 \wedge \omega_0^3 \neq 0$ holds as well as the exterior quadratic equations (17)–(20). The latter equations contain five forms ω_2^3, Δb_{23}, Δb_{33}, ω_1^0, and Δc_3^2 that are different from the basis forms ω_0^2 and ω_0^3. So, we have $q = 5$.

The character s_1 of the system under investigation is equal to the number of independent exterior quadratic equations (17)–(20). Thus, we have $s_1 = 4$. As a result, the second character of the system is $s_2 = q - s_1 = 1$. Therefore, the Cartan number $Q = s_1 + 2s_2 = 6$.

We calculate now the number S of parameters on which the most general integral element of the system under investigation depends (i.e., the dimension S of the space of integral elements over a point). Applying Cartan's lemma to equations (17) and (18), we find that

$$
\begin{cases}
-2b_{23}\omega_2^3 = b_{222}\omega^2 + b_{223}\omega^3, \\
\quad \Delta b_{23} = b_{232}\omega^2 + b_{233}\omega^3, \\
\quad \Delta b_{33} = b_{332}\omega^2 + b_{333}\omega_0^3.
\end{cases}
\tag{22}
$$

Since the coefficients of the basis forms in the right-hand sides of (22) are symmetric with respect to the lower indices, the number of independent among these coefficients is $S_1 = 4$.

Equation (20) implies that

$$
\omega_1^0 = c_3^2 \omega_2^3 + \lambda \omega_0^3.
\tag{23}
$$

We substitute this expression into equation (19). As a result, we obtain

$$
-2(c_3^2 \omega_2^3 + \lambda \omega_0^3) \wedge \omega_0^2 + \Delta c_3^2 \wedge \omega_0^3 = 0.
\tag{24}
$$

It follows from (24) that the 1-form Δc_3^2 is a linear combination of the basis forms. We write this expression in the form

$$
\Delta c_3^2 = \mu \, \omega_0^2 + \nu \, \omega_0^3.
\tag{25}
$$

We can find the form ω_2^3 from the first equation of (22). Substituting this expression and (25) into equation (24), we find that

$$
\left(\frac{c_3^2 b_{223}}{b_{23}} - \lambda \right) \omega_0^3 \wedge \omega_0^2 + \mu \, \omega_0^2 \wedge \omega_0^3 = 0.
$$

This implies that

$$\mu = \frac{c_3^2 b_{223}}{b_{23}} - \lambda.$$

Thus, there are only two independent coefficients in decompositions (23) and (25), $S_2 = 2$. As a result, we have $S = S_1 + S_2 = 6$, and $S = Q$. Thus, by Cartan's test, the system under investigation is in involution, and its general solution depends on one function of two variables.

Next, we find the second fundamental forms of the two-dimensional focal surface G^2 of the hypersurface X with a degenerate Gauss map. To this end, we compute

$$d^2 A_1 \equiv (\omega_1^0 \omega_0^3 + \omega_1^2 \omega_2^3) A_3 + \omega_1^2 \omega_2^4 A_4 \pmod{T_{A_1}(G^2)}.$$

Thus, the second fundamental forms of G^2 are

$$\Phi_1^3 = \omega_1^0 \omega_0^3 + \omega_1^2 \omega_2^3, \quad \Phi_1^4 = \omega_1^2 \omega_2^4.$$

The direction $A_1 A_0$ is defined on G_2 by the equation $\omega_1^2 = 0$. By (16), this equation is equivalent to the equation $\omega_0^3 = 0$. Thus, in this direction the second fundamental forms Φ_1^3 and Φ_1^4 vanish:

$$\Phi_1^3 \equiv 0 \pmod{\omega_0^3}, \quad \Phi_1^4 \equiv 0 \pmod{\omega_0^3},$$

and the direction $A_1 A_0$ is an asymptotic direction on the focal surface G^2. $\square$

2) We prove the following existence theorem for the twisted cones.

Theorem 2. *If condition* (21) *is satisfied, then the double focus A_1 of the generator $A_0 \wedge A_1$ of the variety X describe the focal curve, and X is a twistor cone. In the space P^4, the twisted cones exist, and the general solution of the system defining such cones depends on five functions of one variable.*

Proof. In this case, the point A_1 describes the focal line G^1. Thus we must enlarge the system of equations (16) by the equation

$$\omega_1^0 = a\,\omega_0^3. \tag{26}$$

Equation (26) is equivalent to equation (21). The 1-form ω_0^3 is a basis form on the focal line G^1. By (26), equation (20) takes the form

$$\omega_2^3 \wedge \omega_0^3 = 0.$$

It follows that

$$\omega_2^3 = b\,\omega_0^3. \tag{27}$$

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Now equations (27) and (19) become

$$(\Delta b_{23} + 2b_{23}\, b\, \omega_0^2) \wedge \omega_0^3 = 0, \qquad (28)$$

$$(\Delta c_3^2 + (a + b\, c_3^2)\, \omega^2) \wedge \omega_0^3 = 0. \qquad (29)$$

Equation (28) remains the same.

Taking exterior derivatives of equations (26) and (28), we obtain the exterior quadratic equations

$$(da + a(2\omega_0^0 - \omega_1^1 - \omega_3^3) + c_3^2\, \omega_2^0 + ab\, \omega_0^2) \wedge \omega_0^3 = 0, \qquad (30)$$

$$(db + b(\omega_0^0 - \omega_2^2) + b_{23}\, \omega_4^3 + b\, \omega_0^2) \wedge \omega_0^3 = 0. \qquad (31)$$

Now the system of exterior quadratic equations consists of equations (18), (28)–(31). Thus, we have $s_1 = 5$. In addition to the basis forms ω_0^2 and ω_0^3, these exterior equations contain the forms Δb_{23}, $\Delta b_{33}, \Delta c_3^2, \Delta a$, and Δb, where

$$\Delta a = da + a(2\omega_0^0 - \omega_1^1 - \omega_3^3) + c_3^2\omega_2^0 \qquad (32)$$

and

$$\Delta b = db + b(\omega_0^0 - \omega_2^2) + b_{23}\omega_4^3.$$

The number of these forms is $q = 5$. Thus, $s_2 = q - s_1 = 0$, and the Cartan number $Q = s_1 = 5$. If we find the forms Δb_{23}, $\Delta b_{33}, \Delta c_3^2, \Delta a$, and Δb from the system of equations (18), (28)–(31), we see that the most general integral element of the system under investigation (i.e., the dimension S of the space of integral elements over a point) depends on $S = 5$ parameters. Thus, $S = Q$, the system under investigation is in involution, and its general solution depends on five functions of one variable. $\qquad \square$

Consider the focal curve G^1 of the twisted cone $X^3 \subset P^4$ described by the point A_1. We have

$$dA_1 = \omega_1^1 A_1 + (c_3^2 A_2 + a A_0)\, \omega_0^3.$$

The point $\widetilde{A}_2 = c_3^2 A_2 + a A_0$ along with the point A_1 define a tangent line to G^1. We can specialize our moving frame by locating its vertex A_2 at $\widetilde{A}_2$ and by normalizing the frame by means of the condition $c_3^2 = 1$. Then we obtain

$$dA_1 = \omega_1^1 A_1 + \omega_0^3 A_2.$$

In addition, the conditions

$$a = 0, \quad c_3^2 = 1$$

are satisfied. These conditions and equations (16), (26), (29), and (32) imply that

$$\omega_1^2 = \omega_0^3, \quad \omega_1^0 = 0, \qquad (33)$$

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$$\Delta c_3^2 = \omega_0^0 - \omega_1^1 + \omega_2^2 - \omega_3^3, \tag{34}$$

$$\Delta a = \omega_2^0. \tag{35}$$

After this specialization, the straight line $A_1 \wedge A_2$ becomes the tangent to the focal line G^1.

Now equations (29) and (30) take the form

$$(\omega_0^0 - \omega_1^1 + \omega_2^2 - \omega_3^3 + b\,\omega_0^2) \wedge \omega_0^3 = 0,$$
$$\omega_2^0 \wedge \omega_0^3 = 0.$$

It follows from the last equation that

$$\omega_2^0 = c\,\omega_0^3. \tag{36}$$

Note also that equation (31) shows that since $b_{23} \neq 0$, the quantity b can be reduced to 0 by means of the form ω_4^3. As a result, equation (27) takes the form

$$\omega_2^3 = 0, \tag{37}$$

and equation (31) becomes

$$\omega_4^3 \wedge \omega_0^3 = 0. \tag{38}$$

It follows from (38) that

$$\omega_4^3 = f\,\omega_0^3. \tag{39}$$

Differentiating the point A_2 and applying (16), (36), and (37), we obtain

$$dA_2 = \omega_2^2 A_2 + \omega_2^1 A_1 + (cA_0 + b_{23}A_4)\,\omega_0^3.$$

The 2-plane $\alpha = A_1 \wedge A_2 \wedge (cA_0 + b_{23}A_4)$ is the *osculating plane* of the line G^1 at the point A_1.

We place the point A_4 of our moving frame into the plane α and make a normalization $b_{23} = 1$. As a result, we have $c = 0$ and

$$\omega_2^0 = 0, \quad \omega_2^4 = \omega_0^3. \tag{40}$$

Now, the plane α is defined as $\alpha = A_1 \wedge A_2 \wedge A_4$, and the differential of A_2 becomes

$$dA_2 = \omega_2^2 A_2 + \omega_2^1 A_1 + \omega_0^3 A_4.$$

Taking the exterior derivative of the first equation of (40), we obtain

$$\omega_4^0 \wedge \omega_0^3 = 0,$$

and this implies that

$$\omega_4^0 = g\,\omega_0^3. \tag{41}$$

By means of equations (37) and (41), we find that

$$dA_4 = \omega_4^4 A_4 + \omega_4^1 A_1 + \omega_4^2 A_2 + (fA_3 + gA_0)\,\omega_0^3. \tag{42}$$

Equation (42) means that the 3-plane

$$\beta = A_1 \wedge A_2 \wedge A_4 \wedge (fA_3 + gA_0)$$

is the *osculating hyperplane* of the focal line G^1.

Taking exterior derivatives of equations (39) and (41), we find the following exterior quadratic equations:

$$(df + f(\omega_0^0 - \omega_4^4)) \wedge \omega_0^3 = 0, \tag{43}$$

and

$$(dg + g(2\omega_0^0 - \omega_3^3 - \omega_4^4) - f\omega_3^0) \wedge \omega_0^3 = 0. \tag{44}$$

As we did earlier, we can prove that by means of the secondary forms $\omega_0^0 - \omega_4^4$ and ω_3^0, we can specialize our moving frames in such a way, that

$$f = 1, \quad g = 0.$$

As a result, equations (39) and (41) become

$$\omega_4^3 = \omega_0^3, \quad \omega_4^0 = 0, \tag{45}$$

and the osculating hyperplane β of G^1 becomes

$$\beta = A_1 \wedge A_2 \wedge A_4 \wedge A_3.$$

Substituting the values $f = 1$ and $g = 0$ into equations (43) and (44), we obtain

$$(\omega_0^0 - \omega_4^4) \wedge \omega_0^3 = 0, \tag{46}$$

and

$$\omega_3^0 \wedge \omega_0^3 = 0. \tag{47}$$

Note that equations (46) and (47) could be also obtained by exterior differentiation of equations (45).

After the above specialization, we obtain the following system of equations defining the twisted cones X in the space P^4:

$$\begin{cases} \omega_2^4 = \omega_0^3, & \omega_3^4 = \omega^2, \\ \omega_1^2 = \omega_0^3, & \omega_1^3 = 0, \\ \omega_1^0 = 0, & \omega_2^3 = 0, \\ \omega_2^0 = 0, & \omega_2^4 = \omega^3, \\ \omega_4^3 = \omega_0^3, & \omega_4^0 = 0. \end{cases} \tag{48}$$

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Note that in addition to all specializations made earlier, in equations (48), we also made a specialization $b_{33} = 0$ which can be achieved by means of the secondary form $\omega_0^1 - \omega_3^2$ (see the third equation in (22)).

Taking exterior derivatives of equations (48), we find the following exterior quadratic equations:

$$
\begin{cases}
(\omega_0^0 - \omega_2^2 - \omega_3^3 + \omega_4^4) \wedge \omega_0^3 = 0, \\
(\omega_0^0 - \omega_2^2 - \omega_3^3 + \omega_4^4) \wedge \omega_0^2 + (\omega_0^1 - \omega_3^2) \wedge \omega_0^3 = 0, \\
(\omega_0^0 - \omega_1^1 + \omega_2^2 - \omega_3^3) \wedge \omega_0^3 = 0, \\
(\omega_0^0 - \omega_4^4) \wedge \omega_0^3 = 0, \\
\omega_3^0 \wedge \omega_0^3 = 0.
\end{cases} \tag{49}
$$

The exterior differentiation of the remaining five equations of system (48) leads to identities.

The system of equations (49) is equivalent to the system of equations (18), (28)–(31) from which it is obtained as a result of specializations. For the system of equations (49), as well as for the original system of equations (18), (28)–(31), we have $q = 5$, $s_1 = 5$, $s_2 = 0$, $Q = N = 5$. *The system is in involution, and its general solution exists and depends on five functions of one variable.*

Further we investigate the structure of the fiber cones of a twisted cone $X \subset P^4$. The fiber cones on X are defined by the system

$$
\omega_0^3 = 0. \tag{50}
$$

By (50) and (48), we have

$$
dA_0 = \omega_0^0 A_0 + \omega_0^1 A_1 + \omega_0^2 A_2. \tag{51}
$$

It follows that the plane $A_0 \wedge A_1 \wedge A_2$ is tangent to the fiber cone C along its generator $L = A_0 \wedge A_1$. By (50) and (48), the differential of the point A_2 is

$$
dA_2 = \omega_2^1 A_1 + \omega_2^2 A_2, \tag{52}
$$

and by (50), we also have

$$
dA_1 = \omega_1^1 A_1. \tag{53}
$$

Equations (51), (52), and (53) prove that the tangent plane $\gamma = A_0 \wedge A_1 \wedge A_2$ to the fiber cone C is fixed when the generator $L = A_0 \wedge A_1$ moves along C. It follows that a fiber cone C is simply a pencil of straight lines with center at A_1 located at the plane γ.

Thus, we have proved the following theorem.

Theorem 3. *A twisted cone X in the space P^4 foliates into a one-parameter family of pencils of straight lines whose centers are located on its focal line G^1 and whose planes are tangent to G^1.*

Namely such a picture can be seen in the example of Sacksteder–Bourgain (see [AG 01a]). However, here we proved this theorem for the general case.

Now we prove the converse statement: A general smooth one-parameter family of two-dimensional planes $\gamma(t)$ in the space P^4 forms a three-dimensional twisted cone X. In fact, such a family envelopes a curve G^1, whose point A is the common point of the planes $\gamma(t)$ and $\gamma(t + dt)$, i.e., $A(t) = \gamma(t) \cap \gamma(t + dt)$. The point $A(t)$ and the plane $\gamma(t)$ define a pencil $(A, \gamma)(t)$ of straight lines with center $A(t)$ and plane $\gamma(t)$. The set of these pencils forms a three-dimensional ruled surface X with rectilinear generators L belonging to the pencils $(A, \gamma)(t)$. Moreover, the tangent space $T(X)$ is constant along a rectilinear generator L. Hence the rank of the variety X equals two.

Since the dimension of the Grassmannian $G(2, 4)$ of two-dimensional planes in the space P^4 is equal to six (see [AG 93], Section 1.4, p. 297), one-parameter family of such planes depends on five functions of one variable. This number coincides with the arbitrariness of existence of twisted cones in P^4 that we computed earlier by investigating a system defining a twisted cone (see Theorem 2).

3) Next we find under what condition a twisted cone becomes a twisted cylinder. This condition is equivalent to a condition under which the osculating hyperplane β of the focal curve G^1 is fixed, when the point A_1 moves along G^1. Since $\beta = A_1 \wedge A_2 \wedge A_3 \wedge A_4$ and

$$dA_3 = \omega_3^0 A_0 + \omega_3^1 A_1 + \omega_3^2 A_2 + \omega_3^3 A_3 + \omega_3^4 A_4,$$

the condition in question has the form

$$\omega_3^0 = 0. \tag{54}$$

If we take the fixed osculating hyperplane β of G^1 as the hyperplane at infinity H_∞ of the space P^4, then P^4 becomes an affine space A^4. Then the hypersurface X becomes a twisted cylinder $\widetilde{X}$, which by Theorem 3, *foliates into a one-parameter family of planar pencils of parallel straight lines.* The hypersurface X does not have singularities in the space A^4 and is a *complete smooth noncylindrical hypersurface.*

It is easy to prove the existence of twisted cylinders in the space A^4.

Theorem 4. *Twisted cylinders in the space A^4 exist, and the general solution of the system defining such cylinders depends on four functions of one variable.*

Proof. In fact, a twisted cylinder in A^4 is defined by the system of equations (48) and (54). By (54), the last equation of (49) becomes an identity. Exterior differentiation of (54) leads to an identity too. Thus, in the system of exterior quadratic equations (51), only four equations are

independent. Thus, $s_1 = 4$, and equations (49) contain only four 1-forms that are different from the basis forms. Hence $q = 4$. Therefore, $s_2 = q - s_1 = 0$, $Q = s_1 + 2s_2 = 4$. Equations (49) imply also that $S = 4$. Since $Q = S$, the system is in involution, and its general solution depends on four functions of one variable. $\square$

In conclusion, we indicate a construction defining the general twisted cylinders in the space A^4. Let P^3 be an arbitrary hyperplane in the projective space P^4, and let G^1 be an arbitrary curve in P^3. Consider a family of planes $\gamma(t)$, that are tangent to the curve G^1 but do not belong to P^3, such that two infinitesimally close planes $\gamma(t)$ and $\gamma(t + dt)$ of this family do not belong to a three-dimensional subspace of the space P^4. Then these two planes have only one common point $A(t) = \gamma(t) \cap \gamma(t + dt)$ belonging to G^1, and the planes $\gamma(t)$ form a twisted cone in the space P^4. If we take the hyperplane P^3 as the hyperplane at infinity of P^4, then the space P^4 becomes an affine space A^4, and a twisted cone formed by the planes $\gamma(t)$ becomes a twisted cylinder in A^4. Such a construction was considered by Akivis in his paper [A 87].

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[1]In References we will use the following abbreviations for the review journals: JFM for *Jahrbuch für die Fortschritte der Mathematik*, MR for *Mathematical Reviews*, and Zbl. for *Zentralblatt für Mathematik und ihren Grenzgebiete*.

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